Quantum Finite Automata and Weighted Automata

M. V. Panduranga Rao[1] and V. Vinay[2]

[1] Department of Computer Science and Automation
Indian Institute of Science
Bangalore 560 012
India.
`pandurang@csa.iisc.ernet.in`
[2] PicoPeta Simputers Pvt Ltd
146, 5th Cross, RMV Extn
Bangalore - 560 080
India.
`vinay@picopeta.com` *

Abstract. Quantum finite automata derive their strength by exploiting interference in complex valued probability amplitudes. Of particular interest is the 2-way model of Ambainis and Watrous that has both quantum and classical states (2QCFA) [A. Ambainis and J. Watrous, Two-way finite automata with quantum and classical state, Theoretical Computer Science, 287(1), pp. 299-311, 2002], since it combines the advantage of the power of interference in a constant-sized quantum system with a 2-way head.

This paper is a step towards finding the least powerful model which is purely classical and can mimic the dynamics of quantum phase. We consider weighted automata with the Cortes-Mohri definition of language recognition [C. Cortes and M. Mohri, Context-Free Recognition with Weighted Automata, Grammars 3(2/3), pp. 133-150, 2000] as a candidate model for simulating 2QCFA.

Given any 2QCFA that (i) uses the accept-reject-continue observable, (ii) recognizes a language with one-sided error and (iii) the entries of whose unitary matrices are algebraic complex numbers, we show a method of constructing a weighted automaton over $\mathbb{C}$ that simulates it efficiently.

1 Introduction

Quantum Finite Automata (QFA) have been an area of active research in the recent past, with a lot models being investigated for their power. The first of such models, proposed by Moore and Crutchfield [9] and Kondacs and Watrous [7], is the Measure-Once (MO) 1-way QFA. In this model, the finite automaton has a read-only input tape and a finite set of states Q with $|Q| = k$, a constant.

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We associate a Hilbert space $\mathcal{H}$ with $Q = \{|q\rangle\}$ forming a basis.[3] A state vector of the MO-1QFA is then a vector of unit norm in this space. A series of unitary transformations, each depending on the input symbol, is applied on an initial quantum state as the input tape is scanned from the beginning to the end. The final state vector is a linear superposition of the basis vectors: $|\psi\rangle = \sum_{q \in Q} \alpha_q |q\rangle$, where $\alpha_q \in \mathbb{C}$ and $\sum_{q \in Q} |\alpha_q|^2 = 1$. On reaching the end of the input, a measurement is performed. If the state observed is among those designated as accepting, the input is accepted. Crutchfield and Moore [9] and Brodsky and Pippenger [4] showed that this model accepts only a proper subset of regular languages with bounded error.

A more powerful model, the Measure-Many (MM) 1-way QFA, was proposed by Kondacs and Watrous [7]. In this model, measurements are made after reading every input symbol. Further, Q is partitioned into *accepting, rejecting* and *non-halting* subspaces. Measurements are carried out in such a way that the outcomes correspond to only the subspaces and not the individual states. If an accepting or rejecting subspace is observed, then the input is accepted or rejected respectively. If the outcome is *non-halting*, the computation proceeds from a normalized vector in the non-halting subspace. Even this model has been shown to accept only a subset of regular languages in [7]. A lot of effort has gone into characterizing the languages accepted by this model, cf. [1, 2, 4]. A still more powerful model, the 2-way quantum finite automaton (2QFA) was also proposed by Kondacs and Watrous [7]. This model allows superpositions where the head can be in many positions simultaneously on the input (of length n). Not only can this model recognize all regular languages, but also some non-regular ones like $L_{eq} = \{a^m b^m \mid m \in \mathbb{N}\}$ with bounded error. However, the $O(\log n)$ qubits required to store the position of the head in the "finite" control make this model costly, and goes against the spirit of finite automata.

Seeking to harness the advantages of both quantum states and the ability of the head of the tape to move both ways, Ambainis and Watrous [3] proposed a model intermediate between 1QFA and 2QFA. They showed that this model, called the 2-way finite automata with quantum and classical states (2QCFA), can recognize L_{eq} and $L_{pal} = \{x \in \{a,b\}^* \mid x = x^R\}$, where x^R is the reverse of x, with bounded error in polynomial and exponential time respectively.

The only resource available to 2QCFA is the power to create interference in probability amplitudes. In this paper we look for an existing classical model that can do the same.

In this paper we focus on 2QCFA with the reasonable restriction that the entries of the unitary operators acting on the quantum component are drawn from the field of algebraic numbers. We compare this variant, which we denote by 2QCFA($\mathbb{A}$), with weighted automata.

A weighted automaton is essentially a DFA with the transitions labelled by weights drawn from a semiring in addition to symbols from a finite alphabet.[4]

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[3] See Section 2 for a quick introduction to quantum computing.

[4] See next section for some basics of weighted automata. See also [6, 12] for theory and [8] for applications.

Cortes and Mohri [5] defined a notion of language recognition by weighted automata based on the sum of path-weights labeled by the input string.[5]

The motivation for comparing weighted automata with 2QCFA is that addition of path-weights can naturally capture interference in weighted automata. However, there are potential hitches. To begin with, the measurement operator does not have a parallel in weighted automata. Secondly, moduli of complex numbers can not be obtained in weighted automata. How does one calculate, say, $\sum_i |\alpha_i|^2$? And finally, one has to reconcile the one-sided bounded error notion of language recognition for 2QCFA with the Cortes-Mohri definition of language recognition for weighted automata.

In this paper we show that these problems can be surmounted when simulating 2QCFA that accept with one-sided error and whose unitary matrices have only algebraic complex numbers as entries. In other words, given a 2QCFA($\mathbb{A}$) that recognizes a language L, we show a method of constructing a weighted automaton over the complex semiring that efficiently recognizes L.

This paper is organized as follows. The next section provides some useful definitions and some basics of 2QCFA and weighted automata. Section 3 gives a simulation of 2QCFA($\mathbb{A}$) by weighted automata.

2   Preliminaries

We begin this section with an introduction to quantum computing. For details, please see the text by Neilson and Chuang [10]. A superposition of k states $\{q_0, q_1, \ldots, q_{k-1}\}$ is a vector of unit norm in a k- dimensional Hilbert space $\mathcal{H}$, with $Q = \{|q_0\rangle, |q_1\rangle, \ldots, |q_{k-1}\rangle\}$ serving as a basis of elementary unit vectors. Thus, a linear superposition may be written as $|\psi\rangle = \sum_{i=0}^{k-1} \alpha_i |q_i\rangle$ with each $\alpha_i \in \mathbb{C}$ and $\sum_{i=0}^{k-1} |\alpha_i|^2 = 1$. A unitary operator on $\mathcal{H}$ is a norm preserving linear operator. Application of a unitary operator U on a superposition $|\psi\rangle$ evolves the system to $U|\psi\rangle$. An orthogonal measurement of the system is specified by a set $\{P_i\}$ of operators such that (i) $P_i = P_i^\dagger$ for each i, where $P_i^\dagger$ denotes the adjoint of P_i (ii) $P_i^2 = P_i$ for each i (iii) $P_i P_j = 0$ for $i \neq j$ and (iv) $\sum_i P_i = I$. Measuring a superposition $|\psi\rangle$ through such a set yields i with a probability $||P_i|\psi\rangle||^2$ for each i. The superposition itself collapses to $\frac{1}{||P_i|\psi\rangle||} P_i |\psi\rangle$ for the i that was the yield. The subspaces E_i of Q on which the projectors P_i operate partition Q; such a partition forms an *observable*.

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[5] In general, power of language recognition depends on the semiring used. Cortes and Mohri investigated this power for several semirings and showed weighted automata for recognizing several classes of context free languages.

2.1 Automata with Classical and Quantum States

Formally, a 2QCFA M is a 9 tuple: $M = (Q, S, \Sigma, \Theta, \delta, q_0, s_0, S_{acc}, S_{rej})$ where

- Q and S are finite sets of quantum and classical states respectively.
- Σ is a finite alphabet, $\Gamma = \Sigma \bigcup \{\text{¢}, \$\}$ being the tape alphabet with ¢,$ \notin \Sigma$ as left and right end-markers respectively.
- q_0 and s_0 are the initial quantum and classical states respectively.
- S_{acc}, $S_{rej} \subseteq S$ are the subsets of classical accepting and rejecting states respectively.
- Θ is a two-parameter transition function which governs the evolution of the quantum state. $\Theta(s, \sigma)$, where $s \in S \backslash (S_{acc} \bigcup S_{rej})$ and $\sigma \in \Gamma$, is either a unitary transformation or a measurement.
- δ is a function describing the evolution of the classical state and is defined as follows:
  Unless $\Theta(s, \sigma)$ is a measurement, $\delta(s, \sigma) \in S \times D$, for $D = \{-1, 0, 1\}$ where $-1, 0$, and 1 specify the movement of the tape head.
  If $\Theta(s, \sigma)$ is a measurement, δ also takes the outcome of the measurement into account.

2.2 Weighted Automata

Weighted automata are essentially DFA with weights (drawn, in general, from a semiring;[6] in our case from $\mathbb{C}$) attached to the transitions, in addition to the symbols from a finite alphabet Σ. There is a set of initial and final states. A weighted automaton W associates an input $x \in \Sigma^*$ to an element $W \circ x$ in $\mathbb{C}$ calculated as the sum of products of the weights along every path from the initial to the final states. Cortes and Mohri [5] defined an elegant notion of language recognition in this setting. If the element $W \circ x$ so associated with the input falls in a predefined subset $\mathbb{J}$ of $\mathbb{C}$, we say that the input is accepted by the weighted automaton W. Thus, in addition to the actual evaluation of $W \circ x$, the time complexity of deciding whether x belongs to the language of W also depends on the complexity of deciding membership in $\mathbb{J}$. Under this definition of language recognition, nondeterministic finite automata and probabilistic finite automata can be seen as specific instances of general weighted automata over different semirings.

While most applications use 1-way weighted automata, we define and use a 2-way version. The definition is similar, except that now the machine reads input off a tape and the head can travel in both directions. A 2-way weighted automaton is formally defined as follows.

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[6] A system $(\mathbb{K}, \oplus, \otimes, \bar{0}, \bar{1})$ is a right semiring if:

- $(\mathbb{K}, \oplus, \bar{0})$ is a commutative monoid with $\bar{0}$ as the identity element for $\oplus$.
- $(\mathbb{K}, \otimes, \bar{1})$ is a monoid with $\bar{1}$ as the identity element for $\otimes$.
- $\forall a, b, c \in \mathbb{K}, (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$, and $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$
- $\forall a \in \mathbb{K}, a \otimes \bar{0} = \bar{0} \otimes a = \bar{0}$.

Definition 1. *A 2-way weighted finite automaton over $\mathbb{C}$ is a 7-tuple $W = (S, \Sigma, I, F, \Delta, \lambda, \rho)$ where*

- *S is a finite set of states.*
- *Σ is a finite alphabet, $\Gamma = \Sigma \bigcup \{\mathcal{\c c}, \$\}$ being the tape alphabet.*
- *$I, F \subseteq S$ are sets of initial and final states respectively.*
- *$\Delta \subseteq S \times \Gamma \times \mathbb{C} \times S \times D$ is a finite set of transitions, where $D = \{-1, 0, +1\}$ is the direction of movement of the head on the tape.*
- *$\lambda : I \to \mathbb{C}$ and $\rho : F \to \mathbb{C}$.*

A weighted automaton recognizes languages in the following manner. For a transition $e \in \Delta$ we denote by $w[e]$ its weight and by $i[e]$ its input label. We denote a *path* $e_1 \ldots e_k \in \Delta^*$ by π. $first[\pi]$ is the originating state of the transition e_1 and $last[\pi]$ is the destination state of e_k. The weight of a path π is given by $w[\pi] = w[e_1] \otimes \ldots \otimes w[e_k]$. If Δ is such that the movement of the head is always deterministic,[7] as will be the case in our case, then we can define $x' = scan_W(x)$ as the sequence symbols of the input x to come under the head. It is important to note that since the head is 2-way, the same symbol of the input may occur more than once in $scan_W(x)$. We denote by $P(q, q')$ the set of all paths between states q and q'. If $P(q, q')$ is the set of all paths between states q and q', we say

$$P(I, F) = \bigcup_{q \in I, q' \in F} P(q, q').$$

Then we can also define

$$\Pi(x') = \{\pi \in P(I, F) | label[\pi] = x'\},$$

where $x' = scan_W(x)$ is the sequence symbols of the input x to come under the head. Moreover, for our purpose, a single initial state q_0 suffices: $|I| = 1$.

The input is associated to a weight in $\mathbb{C}$ by the automaton W as

$$W \circ x' = \sum_{\pi \in \Pi(x')} \lambda(first[\pi]) \otimes w[\pi] \otimes \rho(last[\pi]) \tag{1}$$

With this in hand, we can now define language recognition by a weighted automaton.

Definition 2. *Let $\mathbb{J} \subseteq \mathbb{C}$. We say that a string $x \in \Sigma^*$ is $\mathbb{J}$-recognized by the weighted automaton W if $W \circ scan_W(x) \in \mathbb{J}$.*

3 QCFA and Weighted Automata

We begin with some definitions in the context of 2QCFA($\mathbb{A}$) that will be useful in what follows.

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[7] Head movement is deterministic if the transitions are such that at every time step the head moves in the same direction, no matter what state the machine is in.

Definition 3. *If $\Theta(s, \sigma)$ is a unitary transformation for all $\sigma \in \Sigma$, we call $s \in S$ a* unitary state. *We denote the subset of such states by S_u. On the other hand, if $\Theta(s, \sigma)$ is a measurement operation for any $\sigma \in \Sigma$, we call $s \in S$ a* measurement state *and denote the subset of such states by S_m.*

For the quantum part, we will use an obervable such that the outcome of a measurement will tell if the computation halts immediately by accepting or rejecting the input, or if it is to be continued.

To that end, we partition Q into Q_{acc}, Q_{rej} and Q_{nh}, (for acceptance, rejection and continuation of computation respectively) such that $Q_{acc} \cap Q_{rej} = \phi$ and $q_0 \in Q_{nh}$. If E_{acc}, E_{rej} and E_{nh} are the subspaces spanned by states in Q_{acc}, Q_{rej} and Q_{nh} respectively, we use the observable defined by $E_{acc} \oplus E_{rej} \oplus E_{nh}$. If the outcome of a measurement corresponds to Q_{nh}, computation continues from a normalized vector in the subspace spanned by the states of Q_{nh}. The observable for a "final" measurement will not have a non-halting subspace.

We use the one-sided error notion of language acceptance for the 2QCFA($\mathbb{A}$):

Definition 4. *A 2QCFA($\mathbb{A}$) M is said to recognize a language L with bounded one-sided error $\epsilon > 0$ if*

$$
\begin{aligned}
Prob[\ M \ accepts \ x] &= 1 && \forall x \in L \\
&\leq \epsilon && \forall x \notin L
\end{aligned}
$$

We now state the main result of this paper.

Theorem 1. *Given a 2QCFA($\mathbb{A}$) M that recognizes a language L with one-sided error $\epsilon > 0$, there exists a 2-way weighted automaton W that accepts L in time $O(|scan_W x|)$.*

Proof. Given a 2QCFA($\mathbb{A}$) $M_A = (Q_A, S_A, \Sigma, \Theta, \delta_A, q_0, s_0, S_{acc}, S_{rej})$ that accepts a language L with bounded error probability $\epsilon > 0$, we construct the weighted automaton $W = (S_W, \Sigma, q_0^0, F, \Delta, 1, 1)$ that also accepts L. The set F of final states will have the cardinality of $|S_m||Q_{rej}|$, as we will see soon. We will denote an edge in Δ of a weighted automaton by the tuple (s_1, σ, w, s_2, d): in the state s_1, on seeing the symbol σ, the finite control moves to the state s_2, the head by $d \in \{-1, 0, +1\}$, and the weight of this transition is w.

The Construction

1. The set S_W is the union of the disjoint sets Q_P, S_P and F defined as:
   (a) $Q_P = \{q_{j_1}^i\}$ for $0 \leq i \leq |S_A| - 1$ and $0 \leq j_1 \leq |Q_A| - 1$. Thus we have $|S_A|$ copies of Q_{acc}, Q_{rej} and Q_{nh}, indexed by i as Q_{acc}^i, Q_{rej}^i and Q_{nh}^i respectively.
   (b) $S_P = \{s_{j_2}^i\}$ for $0 \leq i \leq |S_A| - 1$ and $0 \leq j_2 \leq |Q_A| - 1$.
   (c) $F = \{f_j^i\}$ for i such that $s_i \in S_m$ and j such that $q_j^i \in Q_{rej}^i$.

2. If $s_i \in S_u$ and $\Theta(s_i, \sigma)$ is a unitary transformation of the form:

$$U : |q_{j_1}\rangle \to \sum_{j_2=0}^{|Q|-1} \alpha_{j_2}^{j_1} |q_{j_2}\rangle$$

for all $q_{j_1} \in Q$, then add the edge $(q_{j_1}^i, \sigma, \alpha_{j_2}^{j_1}, s_{j_2}^i, 0)$ for $0 \le j_1, j_2 \le |Q| - 1$.

3. If $s_i \in S_m$ and $\Theta(s_i, \sigma)$ is a unitary transformation, we proceed as in the case of S_u. For $s_i \in S_m$ and $\sigma \in \Sigma$ such that $\Theta(s_i, \sigma)$ is a measurement, add the following edges:

   - $(q_j^i, \sigma, 0, q_j^i, 0)$ for all $q_j^i \in Q_{acc}^i$,
   - $(q_j^i, \sigma, 1, s_j^i, 0)$ for all $q_j^i \in Q_{nh}^i$, and
   - $(q_j^i, \sigma, e^{\rho_j^i}, f_j^i, 0)$ for all $q_j^i \in Q_{rej}^i$, where $f_j^i \in F$, e is the base of the natural logarithm, and ρ_j^i are $|S_m||Q_{rej}|$ distinct algebraic numbers.

4. If $\delta_A(s_i, \sigma) = (s_{i'}, d)$, for $s_{i'} \in S_A$, then add the edge $(s_{j_1}^i, \sigma, 1, q_{j_1}^{i'}, d)$.

This completes the construction. For the sake of brevity and clarity, we did not spell out the construction for a "final" measurement state. It is similar, except that now there is no Q_{nh}^i.

Some important observations:

1. The head does not move during transitions from states in $\{q_{j_1}^i\}_{j_1=0}^{|Q_A|-1}$ to states in $\{s_{j_2}^i\}_{j_2=0}^{|Q_A|-1}$.

2. The states in the set $\{q_{j_1}^i\}_{j_1=0}^{|Q_A|-1} \cup \{s_{j_2}^i\}_{j_2=0}^{|Q_A|-1}$ may be seen as belonging to the same *block* labelled i. Since S_A is a set of deterministic states, by step 4 of the construction, so is S_P. Thus, at any point in time, the machine W is in the states of the same block.

3. Intra-block transitions are derived from Θ, while inter-block transitions are derived from δ_A.

4. The number of states in the blocks is actually an overkill, since there are also classical states in the 2QCFA($\mathbb{A}$) that do not play a role in the evolution of the quantum part. By our construction, the blocks corresponding to such states will have trivial internal transitions: $(q_{j_1}^i, 1, \sigma, s_{j_1}^i, 0)$.

Suppose $|\psi\rangle = \sum_{j=0}^{k-1} \gamma_j(x')|q_j\rangle$ is the state vector of the 2QCFA($\mathbb{A}$) after reading the input $x' = scan_{M_A}(x)$. Consider a sequence of unitary transforms uninterrupted by measurements. The block structure then essentially simulates (unitary) matrix multiplication. If i is the latest block entered, it is easy to see that $\gamma_j(x') = \sum_{\pi \in \Pi_j^i(x')} w[\pi]$, where

$$\Pi_j^i(x') = \{\pi \in P(q_0^0, s_j^i) | label[\pi] = x'\}.$$

Consider now a computation wherein measurements have occurred without resulting in termination till some point in time. The fact that it has not terminated implies that at every measurement the quantum part collapsed to a

vector in the subspace spanned by Q_{nh}. Recall that if $|\psi\rangle$ is the state vector before measurement, it collapses to $\frac{P_{nh}|\psi\rangle}{\sqrt{\langle\psi|P_{nh}|\psi\rangle}}$ post measurement, where P_{nh} is the projection operator onto Q_{nh}. While step 3 in the construction ensures that the relative amplitudes and phases of the 2QCFA($\mathbb{A}$) are preserved in the simulating weighted automaton, it is not possible to mimic division by the overall normalization factor $\sqrt{\langle\psi|P_{nh}|\psi\rangle}$. These normalization factors can be accumulated over several measurements and clubbed together. This leads to the following lemma.

Lemma 1. *Let* $|\psi\rangle = \sum_{j=0}^{k-1}\gamma_j(x')|q_j\rangle$ *be the state vector of the 2QCFA($\mathbb{A}$) after reading the input* $x' = scan_{M_A}(x)$. *Then,*

$$\gamma_j(x') = 1/\mathcal{P}\sum_{\pi\in\Pi_j^i(x')}w[\pi]$$

where $\Pi_j^i(x') = \{\pi \in P(q_0^0, s_j^i)|label[\pi] = x'\}$ *and* $\mathcal{P}$ *is an overall normalization factor.*

Since the normalization factor is common to all probability amplitudes, it does not pose a problem as far as interference is concerned.

Lemma 2. *Let* $\Pi(x')$ *be the set* $\{\pi \in P(q_0^0, F)|label[\pi] = x'\}$. *Then,*

$$A \circ x' = \sum_{\pi\in\Pi(x')}w(\pi) = 0$$

if and only if $x \in L$.

Proof. If $x \in L$, the 2QCFA($\mathbb{A}$) accepts x with certainty: the probability amplitude for observing any state in Q_{rej} is zero. By the construction and the previous lemma, $w_j^i = 0$ for all $q_j^i \in Q_{rej}^i$. Therefore the sum $\sum_j w_j^i e^{\rho_j^i}$ of weights of all paths from q_0^0 to the final states f_j^i will be zero for all i such that $s_i \in S_m$.

For the other direction, we begin by noting a classic result of Lindemann (see [11]):

Theorem 2. *Given any distinct algebraic numbers* $\phi_1, \phi_2, \ldots, \phi_n$, *the values* $e^{\phi_1}, e^{\phi_2}, \ldots, e^{\phi_n}$ *are linearly independent over the field of algebraic numbers.*

If $x \notin L$, there exists a non-zero amplitude for some state in Q_{rej}. Again by the above lemma, this means that there will be non-zero weights w_j^i on some paths to states in Q_{rej}^i, for some i such that $s_i \in S_m$. Consider any such i. The sum of weights of all paths from q_0^0 to the final states f_j^i in this block is $\sum_j w_j^i e^{\rho_j^i}$, not all w_j^i being zero. Then, by Lindemann's theorem, this sum is not zero.

Thus, if w_j^i are the weights of the paths ending in the rejecting quantum states $q_j^i \in Q_{rej}^i$ of a block corresponding to a measurement state,

$$\sum_j w_j^i e^{\rho_j^i} = 0 \quad \text{if and only if } w_j^i = 0 \text{ for all } w_j^i,$$

and the lemma follows. $\square$

Therefore, checking whether an input x is in L amounts to checking if $A \circ x'$ equals 0. If it does, we accept the string, else we reject it. Hence, the constructed automaton 0-recognizes L. It is significant that the $\mathbb{J}$ is such a small subset of $\mathbb{J}$, as it reduces the membership test to constant time. The blow-up in the size of the machine is also only by a constant factor. So, the total time taken is $O(|x|)$, owing to a constant number of scans. Hence the theorem. $\square$

3.1   Examples

We illustrate the construction for the 2QCFA($\mathbb{A}$) of Ambainis and Watrous [3] that recognizes palindromes. For convenience, a three dimensional quantum part with real amplitudes is used in [3]. Initially, the quantum part is in the state $|q_0\rangle$. In the first scan, on reading an "a", the matrix A is applied and B on a "b". In the second scan, A^{-1} on reading an "a" and B^{-1} on reading a "b" where

$$\mathbf{A} = \frac{1}{5} \begin{pmatrix} 4 & 3 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} \mathbf{B} = \frac{1}{5} \begin{pmatrix} 4 & 0 & 3 \\ 0 & 5 & 0 \\ -3 & 0 & 4 \end{pmatrix}.$$

After completion of the two scans, a measurement is performed. If the input is a palindrome, then q_0 will be observed every time. Otherwise, either q_1 or q_2 will be observed with a small probability.

Two classical states are required for the two scans. Hence, the corresponding weighted automaton has two "non-trivial" blocks, one for each scan by the 2QCFA. The weights in one block are taken from A and B and in the other, from A^{-1} and B^{-1}.

Figure 1 shows the state transition diagram of the weighted automaton simulating the subroutine. In each box, the states to the left belong to Q_P and the transitions going out of them are labelled with probability amplitudes. The states to the right belong to S_P. Since the classical transitions are deterministic in the 2QCFA($\mathbb{A}$) being simulated, so are the ones going out of these states, by construction. Note that a "trivial block", the block corresponding to the classical state of the machine between the two passes, in which the head is restored to the left end-marker, has not been shown in the figure.

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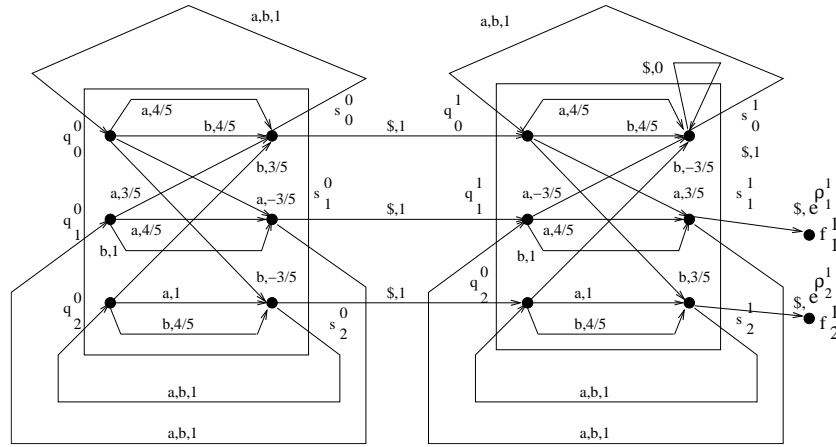
Fig. 1. Recognizing palindromes.

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